Exam Calculus 1

November 28, 2017: 18.30-21.30.

This exam has 8 problems. Each problem is worth 1 point. Total: $8 + 1$ (free) $= 9$ points; more details can be found below. Write on each page your name and student number. The use of annotations, books and calculators is not permitted in this examination. All answers must be supported by work. Success.

1. (a) Formulate the principle of mathematical induction.

   (b) Prove that if $n \geq 1$ is a positive integer, then

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

2. Find all complex solutions z satisfying

$$z^4 - 3z^2 - 1 = 0$$

3. (a) The function f is defined on some open interval that contains the number a, except possibly at a itself. Give the precise definition of

$$\lim_{x \to a} f(x) = L$$

   (b) Prove, using this definition, that

$$\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$$

4. (a) Give the following definition:
   The derivative of a function f at a point a is defined by

$$f'(a) \stackrel{\text{def}}{=} \ldots$$

   (b) We assume that $f(x)$ is differentiable on $-\infty < x < \infty$.
   Give the following definition:
   The derivative f' is continuous at a point a if ...

   (c) A function f is called continuously differentiable on $(-\infty, \infty)$, if $f(x)$ is differentiable for all $x \in (-\infty, \infty)$ and the derivative $f'(x)$ is continuous for all $x \in (-\infty, \infty)$.
   Assume that f and g are continuously differentiable on $(-\infty, \infty)$ and $g'(a) \neq 0$, $f(a) = g(a) = 0$. Show that (l'Hospitals rule is correct):

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

5. (a) Classical mechanics states that the displacement of a particle in a gravitational field is equal to $\frac{1}{2}gt^2$, where t denotes time. According to the theory of relativity the displacement caused by a constant g-force is given by

$$x(t) = \frac{c^2}{g}\left(\left[1 + \left(\frac{gt}{c}\right)^2\right]^{1/2} - 1\right).$$

Here c is the speed of light. Notice: c and g are constant.
Evaluate

$$\lim_{t\to 0} \frac{x(t)}{t^2}.$$

(b) Evaluate

$$\lim_{x\to 0+} x^{\sqrt{x}}$$

6. (a) Show that the values of the following expression do not depend on x:

$$\int_0^x \frac{1}{1+t^2}\,dt + \int_0^{1/x} \frac{1}{1+t^2}\,dt$$

(b) The function f is differentiable on $(-\infty, \infty)$ and

$$\int_0^{x^2} f(t)dt = x^2(1+x)$$

Determine $f(2)$.

7. Evaluate

(a)

$$\int \frac{1}{x \ln x}\,dx$$

(b)

$$\int x\,(\ln x)^2\,dx$$

8. Find all solutions $y(x)$ of the differential equation

(a)

$$xy' - (x+1)y = 0$$

(b)

$$xy' - (x+1)y = x$$

Maximum points:

1a	0.4	2	1.0	3a	0.4	4a	0.2	5a	0.5	6a	0.5	7a	0.5	8	0.5
b	0.6			b	0.6	b	0.2	b	0.5	b	0.5	b	0.5		0.5
						c	0.6								